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The Theory of Spinning Shell

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1.1 Basic Assumption

We assume that the shell is a rigid body. The configuration at any time may therefore be defined by the rectangular coordinates of the center of mass and three Eulerian angles giving the orientation of a set of axes moving with the shell. The state of motion of the shell may be defined by the time rates of change of these six generalized coordinates, making together with them twelve variables capable of independent variation.

We assume that the system of forces and torques acting on the shell, consisting of its weight, and of aerodynamic forces and torques, depends only on these twelve variables, except, perhaps, for small correction terms; more detailed assumptions will be made later.

We assume that the shell has an axis of dynamical symmetry passing through its center of mass.

For simplicity in the following discussion we also neglect the rotation and curvature of the earth, and suppose that there is no wind.

1.2 The Variables Used to Specify the Motion

We use twelve variables, nine of which differ from the above variables.

x,y,z   Firstly we take three rectangular coordinates x, y, and z, of the center of mass of the shell in a fixed coordinate system; the x-direction horizontally forward, the y-direction vertically up. and the z-direction horizontally to the right. The plane x O y contains the line of fire.

v, α, θ
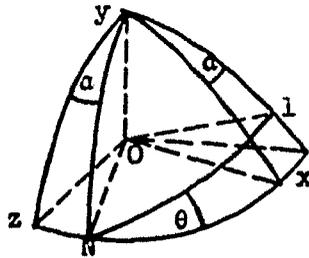
Secondly, we take the speed v of the center of mass and the azimuth α, measured from the x-direction to the left, and angle of elevation θ, measured from the horizontal upwards, of the direction O1 of the velocity of the center of mass, so that we have

O1

$$\left.\begin{array}{l} x = v \cos \theta \cos \alpha \\[4pt] y = v \sin \theta \\[4pt] z = -v \cos \theta \sin \alpha \end{array}\right\}$$

to relate the time rates of change of the coordinates x, y, and z, to these new variables.

(O is not any particular point, but is an origin of reference for directions, which will be represented diagrammatically by points on a sphere of center O.)

ON

ON is to be the direction at angle α forward from Oz, in a horizontal plane, and about which θ is measured.
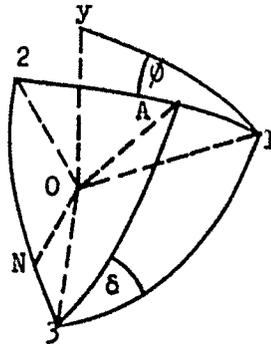
Next complete the determination of the rectangular set of directions O1, O2, and O3, by making the plane 1O2 contain the direction OA of the axis of dynamical symmetry of the shell. Then 1O2 is the plane of yaw, and the angle ϕ from the vertical plane yO1 to the plane 2O1, is ϕ the angle of rotation of the plane of yaw, measured in the righthand direction about O1.

O2, O3
OA

ϕ

O1, O2, and O3, is then a set of rectangular axes with orientation in terms of the fixed directions Ox, Oy, and Oz, given by angles α, θ, and ϕ. Resolving the corresponding angular velocities, $\dot{\alpha}$ about Oy, $\dot{\theta}$ about ON, and $\dot{\phi}$ about O1, in the directions O1, O2, and O3, we obtain the angular velocity components ϑ_1, ϑ_2, and ϑ_3, of this frame of reference referred to itself, namely,

$\vartheta_1, \vartheta_2, \vartheta_3$

$$\vartheta_1 = \dot{\phi} + \dot{\alpha} \sin \theta$$

$$\vartheta_2 = \dot{\alpha} \cos \theta \cos \phi + \dot{\theta} \sin \phi$$

$$\vartheta_3 = -\dot{\alpha} \cos \theta \sin \phi + \dot{\theta} \cos \phi$$

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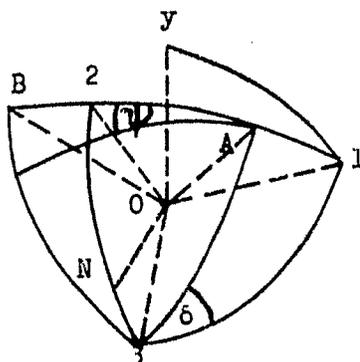
A

We shall use the letter A for the principal moment of inertia of the shell about the axis of dynamical symmetry. On account of the symmetry, the moments of inertia about axes perpendicular to OA through the center of mass are the same;

B

they will be denoted by B.

δ

δ, the angle of yaw is the angle 1OA; we shall call the direction perpendicular to OA in the plane 1OA, (which includes

OB

O2), OB. The principal moments of inertia of the shell about the directions OA, OB, and O3, are then A, B, and B.

ψ

ψ is to be the angle of rotation from the plane AOB to a plane moving with the shell.

We take the angles $\emptyset$, δ, and ψ, as the third set of variables to specify the motion. They are Eulerian angles giving the orientation of the shell relative to the tangent direction O1 to the trajectory of the center of mass and the vertical plane 1Oy through O1.

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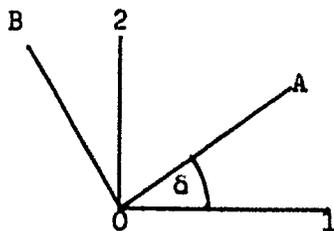
Φ, Δ, Ψ For the fourth set of three variables we take Φ, Δ, and Ψ, the components of angular momentum of the shell about O1, O3, and OA. The components of angular velocity of the shell about OB, O3, and OA, are obtained by resolving ϑ_1, ϑ_2, and ϑ_3, in

these directions, and taking account of the angular velocities corresponding to δ and ψ, giving

$$\vartheta_2 \cos \delta - \vartheta_1 \sin \delta$$

$$\vartheta_3 + \dot{\delta}$$

$$\vartheta_2 \sin \delta + \vartheta_1 \cos \delta + \dot{\psi}$$

Σ The corresponding components of angular momentum, Σ, say, Δ, and Ψ, are

$$\Sigma = B (\vartheta_2 \cos \delta - \vartheta_1 \sin \delta)$$

$$\Delta = B (\vartheta_3 + \dot{\delta})$$

$$\Psi = A (\vartheta_2 \sin \delta + \vartheta_1 \cos \delta + \dot{\psi})$$

so that Φ, the component about O1 is given by

$$\Phi = A \cos \delta(\vartheta_2 \sin \delta + \vartheta_1 \cos \delta + \dot{\psi}) - B \sin \delta(\vartheta_2 \cos \delta - \vartheta_1 \sin \delta).$$

      The variables x, y, z, v, α, θ, ϕ, δ, ψ, Φ, Δ, and Ψ, are all given if the configuration and velocity of the shell are given. This would not be true if we tried to use $\dot{\phi}$, $\dot{\delta}$, and $\dot{\psi}$, for these would involve $\dot{\alpha}$ and $\dot{\theta}$, and therefore $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$.

      The component of angular momentum in the direction O2 is

$$B \cos \delta(\vartheta_2 \cos \delta - \vartheta_1 \sin \delta) + A \sin \delta(\vartheta_2 \sin \delta + \vartheta_1 \cos \delta + \dot{\psi})$$

$$= \frac{\Psi - \Phi \cos \delta}{\sin \delta} ,$$

while

$$\Sigma = \frac{\Psi \cos \delta - \Phi}{\sin \delta}$$

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1.3 The Equations of Motion

The absolute rate of change of a vector whose components in the O1, O2, and O3 directions are L_1, L_2, and L_3 has components

$$\dot{L}_1 + \vartheta_2 L_3 - \vartheta_3 L_2$$

$$\dot{L}_2 + \vartheta_3 L_1 - \vartheta_1 L_3$$

and

$$\dot{L}_3 + \vartheta_1 L_2 - \vartheta_2 L_1$$

M

If M is the mass, the linear momentum has components in these directions Mv, 0, and we have the equations

$$\left. \begin{array}{r} M\,\dot{v} = F_1 \\ \vartheta_3 \, M \, v = F_2 \\ -\vartheta_2 \, M \, v = F_3 \end{array} \right\}$$

F_1, F_2, F_3

for the rates of change of momentum equated to the components of force acting on the shell in the O1, O2, and O3 directions.

In the same way, the components of angular momentum being

$$\Phi, \; \frac{\Psi - \Phi \cos \delta}{\sin \delta}, \; \text{and } \Delta, \; \text{we obtain}$$

$$\left. \begin{array}{r} \dot{\Phi} + \vartheta_2 \Delta - \vartheta_3 \left(\dfrac{\Psi - \Phi \cos \delta}{\sin \delta} \right) = G_1 \\[2mm] \dfrac{d}{dt} \left(\dfrac{\Psi - \Phi \cos \delta}{\sin \delta} \right) + \vartheta_3 \Phi - \vartheta_1 \Delta = G_2 \\[2mm] \dot{\Delta} + \vartheta_1 \left(\dfrac{\Psi - \Phi \cos \delta}{\sin \delta} \right) - \vartheta_2 \Phi = G_3 \end{array} \right\}$$

$G_1, \; G_2, \; G_3$ in terms of the components of torque about its center of mass on the shell in the O1, O2, and O3 directions.

It is usual, however, to regard the torque components about the directions OA, OB, and O3, namely G_A, G_B, and G_3, as primary, and to express the equations of motion in terms of them.

$$G_A = G_1 \cos \delta + G_2 \sin \delta$$

$$G_B = -G_1 \sin \delta + G_2 \cos \delta$$

Combining the first two of these equations we obtain

$$\dot{\Psi} = G_2 \sin \delta + G_1 \cos \delta = G_A \quad,$$

the remaining terms cancelling, which equation can be obtained directly from the absolute rate of change of angular momentum about OA.

Eliminating ϑ_1, ϑ_2, and ϑ_3, we find from these equations and from the equations of the last section, equations for the rates of change of our twelve variables in terms of these variables and of the components of force F_1, F_2, and F_3, and of torque G_A, G_B, and G_3.

$$\dot{x} = v \cos \theta \cos \alpha$$

$$\dot{y} = v \sin \theta$$

$$\dot{z} = -v \cos \theta \sin \alpha$$

$$\dot{v} = \frac{F_1}{M}$$

$$\dot{\alpha} = -\frac{F_2}{Mv} \frac{\sin \phi}{\cos \theta} - \frac{F_3}{Mv} \frac{\cos \phi}{\cos \theta}$$

$$\dot{\theta} = \frac{F_2}{Mv} \cos \phi - \frac{F_3}{Mv} \sin \phi$$

$$\dot{\delta} = \frac{1}{B} \Delta - \frac{F_2}{Mv}$$

$$\dot{\phi} = \frac{\Phi - \Psi \cos \delta}{B \sin^2 \delta} + \frac{F_2}{Mv} \sin \phi \tan \theta + \frac{F_3}{Mv} (\cos \phi \tan \theta - \cot \delta)$$

$$\dot{\Psi} = \frac{1}{A} \Psi - \frac{\cos \delta}{B \sin^2 \delta} (\Phi - \Psi \cos \delta) + \frac{F_3}{M v \sin \delta}$$

$$\dot{\Delta} = - \frac{(\Phi - \Psi \cos \delta)(\Psi - \Phi \cos \delta)}{B \sin^3 \delta} + G_3 - \frac{(\Phi - \Psi \cos \delta)}{\sin^2 \delta} \frac{F_3}{Mv}$$

$$\dot{\Phi} = G_A \cos \delta - G_B \sin \delta + \Delta \frac{F_3}{Mv} + \frac{(\dot{\Psi} - \dot{\Phi} \cos \delta)}{\sin \delta} \; \frac{F_2}{Mv} \Bigg\}$$

$$\dot{\Psi} = G_A$$

Part II.   The Forces and Torques

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2.1 The force and Torque Components
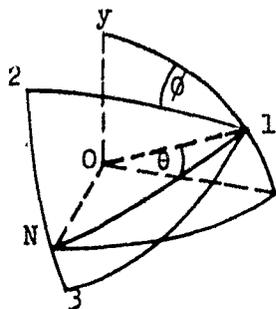
The force system acting on the shell is made up of gravitational forces and of aerodynamic forces and torques. Gravitation contributes terms $-M g \sin \theta$, $-M g \cos \theta \cos \phi$, and $M g \cos \theta \sin \phi$ to F_1, F_2, and F_3 respectively.
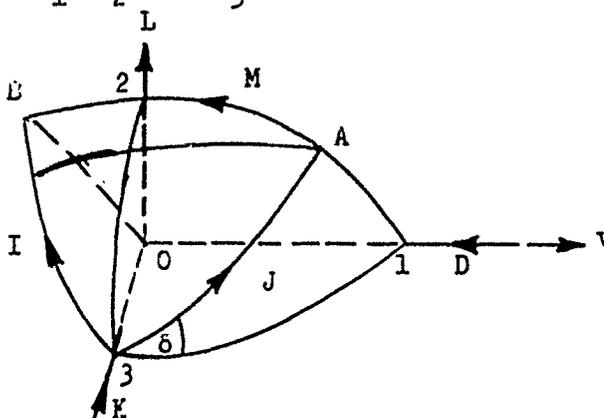
The aerodynamic forces are defined relative to the directions O1, O2, and O3, but are taken positive in their expected
directions, the drag D in the opposite direction to O1, the
lift or cross-wind force L in the direction O2, and the Magnus
force K in the opposite direction to O3. For a symmetrical
projectile rotating from OB to O3 about O1 we should expect a
force in this direction; a ball swerving on account of spin
follows its nose.

D
L
K

Thus
$$F_1 = -D - mg \sin \theta$$
$$F_2 = L - mg \cos \theta \cos \phi$$
$$F_3 = -K + mg \cos \theta \sin \phi$$

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The torques are defined relative to the directions OA, OB, and O3, but are taken positive in their expected direc-
I   tions; the spin destroying or rolling moment I from O3 to OB;
M   the yawing or overturning moment M from OA to OB; and the
J   Magnus moment J from O3 to OA. For a projectile with Magnus
force in the opposite direction to O3 acting in front of the center of mass, we should expect a torque in this direction.

Thus
$$\left.\begin{array}{l} G_A = -I \\[2mm] G_B = J \\[2mm] G_3 = M \end{array}\right\}$$

2.2 Several Assumptions About the Aerodynamic Forces and Torques

We assume that the aerodynamic forces and torques depend only on the properties of air through which the shell is pass-ing, which we shall suppose depends only on y, and on the motion of the shell relative to the air; that is, they do not depend on x, z, α, θ, or $\emptyset$. Thus the aerodynamic forces and torques

_depend only on y, v, δ, ψ, $\frac{\Psi}{A}$, $\frac{\Sigma}{B}$, $\frac{\Delta}{B}$, in a manner de-
termined by the properties of the air, specified by its density
ρ,c,μ ρ, sound velocity c, and viscosity μ; the size of the shell,
d   specified by its caliber d, and its shape relative to its center of mass.

Here we have tacitly assumed that any effects of lag in the distribution of airflow can be taken account of by the
dependence on $\frac{\Psi}{A}$, $\frac{\Sigma}{B}$, and $\frac{\Delta}{B}$.

Dynamical similarity then requires that force components be proportional to $\rho\, v^2\, d^2$ and torque components to $\rho\, v^2\, d^3$, with coefficients depending on the zero-dimensional quantities
$\mathcal{M},R$ v/c = $\mathcal{M}$, the Mach number, $\frac{\mu}{\rho\, v\, d}$ = R, the Reynolds number, the
number
of radians turned per caliber advanced, $\frac{d}{v}\frac{\Psi}{A}$, $\frac{d}{v}\frac{\Sigma}{B}$, and

$\frac{d}{v}\frac{\Delta}{B}$, and the angles δ and ψ.

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2.3 Aerodynamic Forces and Torques for a Shell with Rotational Symmetry

For a shell with rotational symmetry about its axis, the aerodynamic force and torque components will not depend on ψ. Change of sign of δ, Δ, and Σ, equivalent to increase of ϕ by π and reduction of ψ by π, must change the signs of L, K, J, and M, leaving D and I unaltered; while change in sign of Σ and ψ, equivalent to inversion in the origin, must change the signs of K, I, and J, leaving M, D, and L unaltered. Thus if any of these components are split up into terms even and odd in Δ and Σ, the evenness or oddness of each term in ψ and δ is determined, and we may write:

$D =$

$K_D, K_{D\Delta}$

$$\rho v^2 d^2 K_D + \rho v d^3 \sin \delta \frac{\Delta}{B} K_{D\Delta} + \rho d^4 \sin \delta \frac{\Sigma}{B} \frac{\psi}{A} K_{D\Sigma}$$

$K_{D\Sigma}, K_{D\Delta\Sigma}$

$$+ \rho \frac{d^5}{v} \frac{\Delta}{B} \frac{\Sigma}{E} \frac{\psi}{A} K_{D\Delta\Sigma}$$

$L =$

$K_L, K_{L\Delta}$

$$\rho v^2 d^2 \sin \delta K_L + \rho v d^3 \frac{\Delta}{B} K_{L\Delta} + \rho d^4 \frac{\Sigma}{B} \frac{\psi}{A} K_{L\Sigma}$$

$K_{L\Sigma}, K_{L\Delta\Sigma}$

$$+ \rho \frac{d^5}{v} \sin \delta \frac{\Delta}{B} \frac{\Sigma}{B} \frac{\psi}{A} K_{L\Delta\Sigma}$$

$K =$

$K_K, K_{K\Delta}$

$K_{K\Sigma}, K_{K\Delta\Sigma}$

$$\rho v d^3 \sin \delta \frac{\psi}{A} K_K + \rho d^4 \frac{\Delta}{B} \frac{\psi}{A} K_{K\Delta} + \rho d^3 v \frac{\Sigma}{B} K_{K\Sigma}$$

$$+ \rho d^4 \sin \delta \frac{\Delta}{B} \frac{\Sigma}{B} K_{K\Delta\Sigma}$$

$I =$

$K_I, K_{I\Delta}$

$$\rho v d^4 \frac{\psi}{A} K_I + \rho d^5 \sin \delta \frac{\Delta}{B} \frac{\psi}{A} K_{I\Delta} + \rho d^4 v \sin \delta \frac{\Sigma}{B} K_{I\Sigma}$$

$K_{I\Sigma}, K_{I\Delta\Sigma}$

$$+ \rho d^5 \frac{\Delta}{B} \frac{\Sigma}{B} K_{I\Delta\Sigma}$$

$K_J, K_{J\Delta}$

$$J =$$

$$\rho v d^4 \sin \delta \frac{\Psi}{A} K_J + \rho d^5 \frac{\Delta}{B} \frac{\Psi}{A} K_{J\Delta} + \rho d^4 v \frac{\Sigma}{B} K_{J\Sigma}$$

$$+ \rho d^5 \sin \delta \frac{\Delta}{B} \frac{\Sigma}{B} K_{J\Delta\Sigma}$$

$K_{J\Sigma}, K_{J\Delta\Sigma}$

$$M =$$

$K_M, K_{M\Delta}$

$$\rho v^2 d^3 \sin \delta K_M + \rho v d^4 \frac{\Delta}{B} K_{M\Delta} + \rho d^5 \frac{\Sigma}{B} \frac{\Psi}{A} K_{M\Sigma}$$

$$+ \rho \frac{d^6}{v} \sin \delta \frac{\Delta}{B} \frac{\Sigma}{B} \frac{\Psi}{A} K_{M\Delta\Sigma}$$

$K_{M\Sigma}, K_{M\Delta\Sigma}$

where the coefficient K_D, ---- $K_{M\Delta\Sigma}$, are functions of M and R, even in $\frac{d}{v} \frac{\Psi}{A}$, $\frac{d}{v} \frac{\Sigma}{B}$, $\frac{d}{v} \frac{\Delta}{B}$, and δ, and are thereby uniquely determined. (Note that use of sin δ rather than δ allows δ to pass through the value π as well as the value zero, our statements remaining valid; but it would not be true to say that the coefficients are functions of sin δ as they will usually differ at $\delta = 0$ and $\delta = \pi$).

Further, when sin $\delta = 0$, D and I can depend on Σ and Δ only through $\Sigma^2 + \Delta^2$, so that K_D and K_I will depend on $\Sigma^2 + \Delta^2$ and $K_{D\Delta\Sigma}$ and $K_{I\Delta\Sigma}$ will vanish, while L and K, and J and M must behave like the components of plane vectors depending on the vector with components Σ and Δ, so that $K_{L\Sigma} =$

K_{XF}, K_S

$-K_{K\Delta}$, $K_{L\Delta} = K_{K\Sigma}$, $- K_{J\Delta} = K_{M\Sigma}$, and $K_{J\Sigma} = K_{M\Delta}$, being what

K_{XT}, K_H

are commonly called K_{XF}, K_S, K_{XT}, and K_H, and likewise must

be functions of $\Sigma^2 + \Delta^2$ only. In general these are all different functions of $\Sigma^2 + \Delta^2$ for $\delta = 0$ and for $\delta = \pi$.

It must be emphasized, however, that when sin $\delta \neq 0$, these last relations do not have to hold, although the departure from them must be of order $\sin^2 \delta$.

Part III.   Motion of a Spinning Symmetrical
            Projectile with Large Yaw

3.1 The General Method of Transformation by Variation of Parameters

A solution is first made of a reduced problem in which many terms in the rates of change of the variables have been left out. This leads to a transformation from the variables to a new set, the parameters, which would be constant in time if the retained terms alone existed. Applying this transformation to the original equations, new exact equations are obtained for the rates of change of the parameters, which change more slowly than the original variables.

If both the original equations and the reduced equations are of Hamiltonian form, the transformation may be carried out on the Hamiltonian function only. If the reduced equations are of Hamiltonian form, it may still be convenient to solve them by the Hamilton-Jacobi method, although the transformations resulting have to be carried out on the separate equations instead of on a single function.

If the reduced equations can be taken linear, we have the usual method of variation of parameters, which, for a second order system, leads to the W. K. B. method so much used in quantum mechanics.

If the solution of the reduced problem is periodic with a comparatively short period, the next step usual in the solution is to approximate the equations of change of the parameters by averaging over a period. This gives an approximate system of 'secular equations' for the change of the parameters, which represent the motion well, unless there is resonance with some of the ignored periodic terms.

If in this last case the reduced equations are in Hamiltonian form, the general increase or decrease in amplitude of the periodic motion may be tested by the average change in the Hamiltonian function itself, the standard 'energy test of stability'.

It is this last case that we are going to consider; so we are ruling out possible cases where very large damping may make stable motion that would be unstable for small damping.

3.2 The Reduced Equations and Their Solution

The reduced equations

$$\dot{\delta} = \frac{1}{B}\,\Delta$$

$$\dot{\phi} = \frac{\Phi - \Upsilon \cos\delta}{B \sin^2 \delta}$$

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$$\dot{\psi} = \frac{1}{A}\, \Psi - \frac{\cos \delta}{B \sin^2 \delta}\, (\Phi - \Psi \cos \delta)$$

$$\dot{\Delta} = -\frac{(\Phi - \cos \delta)(\Psi - \Phi \cos \delta)}{B \sin^3 \delta} - \frac{\partial}{\partial \delta}\, V(\delta)$$

$$\dot{\Phi} = 0$$

$$\dot{\Psi} = 0$$

V(δ)

give the motion in angle when all the force and torque com-
ponents except the overturning moment (M or G_3) are neglected,
and when that is regarded as derived from a potential V(δ).
These equations correspond to a Hamiltonian function

H

$$H = \frac{1}{2B}\left\{ \Delta^2 + \frac{1}{\sin^2 \delta}\, (\Phi - \Psi \cos \delta)^2 \right\} + \frac{1}{2A}\, \Psi^2 + V(\delta) \; ,$$

just that for a symmetrical top with potential energy V(δ).

The Hamilton-Jacobi equation

S

$$\frac{\partial S}{\partial t} + \frac{1}{2B}\left\{ \left(\frac{\partial S}{\partial \delta}\right)^2 + \frac{1}{\sin^2 \delta}\left(\frac{\partial S}{\partial \phi} - \frac{\partial S}{\partial \psi}\cos \delta\right)^2 \right\} + \frac{1}{2A}\left(\frac{\partial S}{\partial \psi}\right)^2$$
$$+ V(\delta) = 0$$

has the complete integral

S_0

$$S = S_0 - Ht + \Phi_0\, \phi + \Psi_0\, \psi + \int_{\delta_0}^{\delta}\left\{ 2B\left(H - \frac{1}{2A}\, \Psi^2 - V(\delta)\right)\sin^2 \delta \right.$$
$$\left. - (\Phi_0 - \Psi_0 \cos \delta)^2 \right\}^{\frac{1}{2}} \frac{d\delta}{\sin \delta} \; ,$$

Φ_0, Ψ_0 and the solutions of the reduced equations are given by

ϕ_0, Ψ_0, δ_0 $\quad \Delta = \dfrac{\partial S}{\partial \delta} \; , \; \Phi = \dfrac{\partial S}{\partial \phi}, \; \Psi = \dfrac{\partial S}{\partial \psi} \; , \; t_0 = \dfrac{-\partial S}{\partial H} \; , \; \phi_0 = \dfrac{\partial S}{\partial \Phi_0} \; , \; \Psi_0 = \dfrac{\partial S}{\partial \Psi_0} \; ,$

where ϕ_0, Ψ_0, and δ_0, as well as Φ_0 and Ψ_0, correspond to time t_0.

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Since ϕ and ψ do not occur explicitly in H, $\dot{\Phi}$ and $\dot{\Psi}$ are equal to their initial values $\dot{\Phi}_0$ and $\dot{\Psi}_0$. The motion in δ is in general a libration between two simple roots of

$$2B \left(H - \frac{1}{2A} \dot{\Psi}_0^2 - V(\delta) \right) \sin^2 \delta - (\dot{\Phi}_0 - \dot{\Psi}_0 \cos \delta)^2 = 0$$

between which this expression is positive.

δ_0 It is convenient to take δ_0 to be the smaller of these roots, simplifying the partial derivatives of S. We call the larger root
δ_1 δ_1.

The solution is then given by

$$\Delta = \frac{\left\{ 2B \left(H - \frac{1}{2A} \dot{\Psi}_0^2 - V(\delta) \right) \sin^2 \delta - (\dot{\Phi}_0 - \dot{\Psi}_0 \cos \delta)^2 \right\}^{\frac{1}{2}}}{\sin \delta}$$

$$\dot{\Phi} = \dot{\Phi}_0$$

$$\dot{\Psi} = \dot{\Psi}_0$$

$$t_0 = t - \int_{\delta_0}^{\delta} \frac{B \sin \delta \, d\delta}{\left\{ 2B \left(H - \frac{1}{2A} \dot{\Psi}_0^2 - V(\delta) \right) \sin^2 \delta - (\dot{\Phi}_0 - \dot{\Psi}_0 \cos \delta)^2 \right\}^{\frac{1}{2}}}$$

$$\phi_0 = \phi - \int_{\delta_0}^{\delta} \frac{(\dot{\Phi}_0 - \dot{\Psi}_0 \cos \delta)}{\left\{ 2B \left(H - \frac{1}{2A} \dot{\Psi}_0^2 - V(\delta) \right) \sin^2 \delta - (\dot{\Phi}_0 - \dot{\Psi}_0 \cos \delta)^2 \right\}^{\frac{1}{2}}} \frac{d\delta}{\sin \delta}$$

$$\psi_0 = \psi - \int_{\delta}^{\delta} \frac{\frac{B}{A} \dot{\Psi}_0 \sin^2 \delta - (\dot{\Phi}_0 - \dot{\Psi}_0 \cos \delta) \cos \delta}{\left\{ 2B \left(H - \frac{1}{2A} \dot{\Psi}_0^2 - V(\delta) \right) \sin^2 \delta - (\dot{\Phi}_0 - \dot{\Psi}_0 \cos \delta)^2 \right\}^{\frac{1}{2}}} \frac{d\delta}{\sin \delta}$$

since the terms in $d\delta_0$ vanish.

3.3 The Uniformising Variable

For manipulating the above integrals it is convenient to introduce a uniformising variable u which continually increases as δ librates. Using $\cos \delta$ rather than δ as the variable to be given in terms of u, we put

$$\cos \delta = \cos \delta_0 \cos^2 \tfrac{1}{2}u + \cos \delta_1 \sin^2 \tfrac{1}{2}u$$

so that $\delta = \delta_0$ for $u = 0$, $\delta = \delta_1$ for $u = \pi$, and so on.

Then u is given in terms of δ by

$$\tan \frac{u}{2} = \sqrt{\frac{\cos \delta_0 - \cos \delta}{\cos \delta - \cos \delta_1}}$$

with positive sign when δ is increasing, negative sign when δ is diminishing.

Thus

$$\frac{\sin \delta \, d\delta}{\left\{(\cos \delta_0 - \cos \delta)(\cos \delta - \cos \delta_1)\right\}^{\frac{1}{2}}} = du$$

and if

$$\left\{2B\left(H - \frac{1}{2A}\Psi_0{}^2 - V(\delta)\right)\sin^2\delta - (\Phi_0 - \Psi_0\cos\delta)^2\right\} =$$

$$(\cos \delta_0 - \cos \delta)(\cos \delta - \cos \delta_1) \, K(\delta)$$

so that, by the definitions of δ_0 and δ_1, $K(\delta)$ does not vanish

for $\delta_0 \leqslant \delta \leqslant \delta_1$, we have

$$t_0 = t - \int_0^u \frac{B \, du}{\sqrt{K(\delta)}}$$

The standard astronomical proceeding is to expand $1/\sqrt{K(\delta)}$ in a Fourier series in u and so obtain a series which can be inverted to give u in terms of t, although most computations may be more easily carried out in terms of the intermediate variable u.

If we write $\delta_0 = \left|\lambda - \epsilon\right|$, $\delta_1 = \lambda + \epsilon$, so that

$$\tan \frac{u}{2} = \sqrt{\frac{\cos(\lambda - \epsilon) - \cos \delta}{\cos \delta - \cos(\lambda + \epsilon)}}$$

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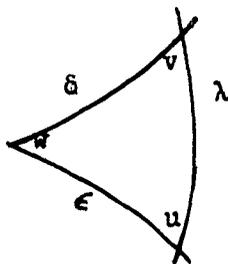
we see that u is the angle included by sides ϵ and λ of a spherical triangle with sides ϵ, λ, and δ. (See appendix.)

Let $\quad \tan \dfrac{v}{2} = \sqrt{\dfrac{\cos(\delta - \lambda) - \cos \epsilon}{\cos \epsilon - \cos(\delta + \lambda)}}$

and $\quad \tan \dfrac{w}{2} = \sqrt{\dfrac{\cos(\epsilon - \delta) - \cos \lambda}{\cos \lambda - \cos(\epsilon + \delta)}}$

so that v and w are the angles included by λ and δ, and by δ and ϵ, while

$$\frac{du}{d\delta} = \frac{\sin \delta}{\left\{(\cos(\epsilon - \lambda) - \cos \delta)(\cos \delta - \cos(\epsilon + \lambda))\right\}^{\frac{1}{2}}}$$

$$\frac{dv}{d\delta} = \frac{\cos \delta \cos \epsilon - \cos \lambda}{\sin \delta \left\{(\cos(\epsilon - \lambda) - \cos \delta)(\cos \delta - \cos(\epsilon + \lambda))\right\}^{\frac{1}{2}}}$$

$$\frac{dw}{d\delta} = \frac{\cos \delta \cos \lambda - \cos \epsilon}{\sin \delta \left\{(\cos(\epsilon - \lambda) - \cos \delta)(\cos \delta - \cos(\epsilon + \lambda))\right\}^{\frac{1}{2}}}$$

so that

$$\phi_o = \phi - \int \frac{\left\{(\Phi_o \cos \lambda - \Psi_o \cos \epsilon) \, dv - (\Phi_o \cos \epsilon - \Psi_o \cos \lambda) \, dw\right\}}{(\cos^2 \epsilon - \cos^2 \lambda) \sqrt{K(\delta)}}$$

$$\psi_o = \psi - \int \frac{\left\{\left(\frac{B}{A} - 1\right) \Psi_o \, du + (\Psi_o \cos \lambda - \Phi_o \cos \epsilon) \, dv - (\Psi_o \cos \epsilon - \Phi_o \cos \lambda) \, dw\right\}}{(\cos^2 \epsilon - \cos^2 \lambda) \sqrt{K(\delta)}}$$

The square roots in the expressions for $\tan \dfrac{u}{2}$, $\tan \dfrac{v}{2}$, and $\tan \dfrac{w}{2}$, and for $\dfrac{du}{d\delta}$, $\dfrac{dv}{d\delta}$, and $\dfrac{dw}{d\delta}$, are to be taken positive when δ is increasing, negative when δ is diminishing, and $\sqrt{K(\delta)}$ is to be taken always positive.

There are now two cases.

(a) $\epsilon < \lambda$.
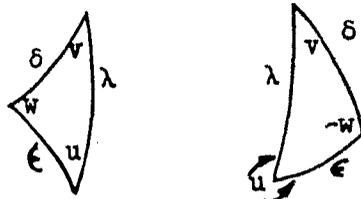
As δ increases from $\lambda - \epsilon$ to $\lambda + \epsilon$, u increases from 0
to π, v increases from 0 to a maximum and falls back to 0, and
w decreases from π to 0: as δ decreases from $\lambda + \epsilon$ to $\lambda - \epsilon$ again,
u increases from π to 2π, v decreases from 0 to a minimum and
rises back to 0, and w decreases from 0 to - π; and so on, u,
the uniformising variable, continually increases, v oscillates,
and w continually decreases.

(b) $\epsilon > \lambda$
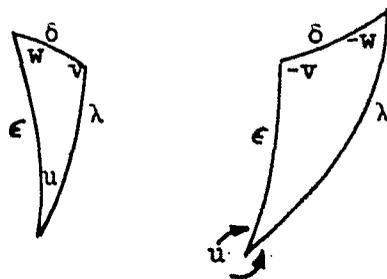
As δ increases from $\epsilon - \lambda$ to $\lambda + \epsilon$, u increases from
0 to π, v decreases from π to 0, and w increases from 0 to a
maximum and falls back to 0; as δ decreases from $\lambda + \epsilon$ to $\epsilon - \lambda$
again, u increases from π to 2π, v decreases from 0 to - π,
and w decreases from 0 to -π, and w decreases from 0 to a
minimum and rises back to 0; and so on, u continually increase,
v continually decreases, and w oscillates.

In the boundary case of $\epsilon = \lambda$, δ passes through the value
0. As δ increases from 0 to $\epsilon + \lambda$, u increases from 0 to π,
v decreases from $\frac{\pi}{2}$ to 0, and w decreases from $\frac{\pi}{2}$ to 0; as δ

decreases from $\epsilon + \lambda$ to 0 again, u increases from π to 2π, v
decreases from 0 to $-\frac{\pi}{2}$, and w decreases from 0 to $-\frac{\pi}{2}$;

as δ passed through the value 0, u continues to increase, and
v and w may be supposed to jump to the values $\frac{\pi}{2}$ and $\frac{\pi}{2}$ again,

without change in those functions of u, v, and w that give
direction cosines of axes in the projectile.

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Physically, λ is a mean yaw, ϵ an amplitude of nutation.

3.4 The Transformation to the Parameters

We now use the above equations to give a transformation from the variables Δ, Φ, Ψ, δ, ϕ, and ψ, to the parameters H, Φ_0, Ψ_0, t_0, ϕ_0, and ψ_0, or to H, λ, ϵ, t_0, ϕ, and ψ_0

evaluated at each value of t, so that V may be allowed to vary with t as well as with δ.

The equations

$$V(\delta,t) \qquad H = \frac{1}{2B}\left\{\Delta^2 + \frac{1}{\sin^2\delta}\ (\Phi - \Psi\cos\delta)^2\right\} + \frac{1}{2A}\Psi^2 + V(\delta,t)$$

$$\Phi_0 = \Phi$$

$$\Psi_0 = \Psi$$

show that we need not distinguish between Φ_0 and Φ and between Ψ_0 and Ψ, and, $\dot{H} = \frac{1}{B}\Delta\dot{\Delta} + \frac{1}{B\sin^2\delta}\ (\Phi - \Psi\cos\delta)(\dot{\Phi} - \dot{\Psi}\cos\delta)$

$$+ \frac{1}{A}\Psi\dot{\Psi} - \frac{1}{B}\frac{\cos\delta}{\sin^3\delta}(\Phi - \Psi\cos\delta)^2\dot{\delta} + \frac{1}{B\sin^2\delta}(\Phi - \Psi\cos\delta)\Psi\sin\delta\ \dot{\delta} + \frac{\partial V}{\partial\delta}\dot{\delta} + \frac{\partial V}{\partial t}\ .$$

Thus

$$\dot{H} = \frac{\Delta}{B}\left(G_3 + \frac{\partial V}{\partial\delta}\right) - \frac{1}{B\sin\delta}\ (\Phi - \Psi\cos\delta)\ G_B + \frac{\Psi}{A}\ G_A - \frac{\partial V}{\partial\delta}\frac{F_2}{Mv} + \frac{\partial V}{\partial t}$$

while $\dot{\Phi} = G_A\cos\delta - G_B\sin\delta + \Delta\frac{F_3}{Mv} + \frac{(\Psi - \Phi\cos\delta)}{\sin\delta}\frac{F_2}{Mv}$

, and $\dot{\Psi} = G_A$

while

$$\dot{H} = \frac{1}{B\sin^2\delta_0}\ (\Phi - \Psi\cos\delta_0)(\dot{\Phi} - \dot{\Psi}\cos\delta) + \frac{1}{A}\Psi\dot{\Psi}$$

$$+\left(\frac{\partial V}{\partial t}\right)_{\delta_0} + \left(-\frac{1}{B}\frac{\cos\delta_0}{\sin^3\delta}(\Phi - \Psi\cos\delta_0)^2 + \frac{1}{B\sin^2\delta_0}(\Phi - \Psi\cos\delta_0)\Psi\sin\delta_0 + \left(\frac{\partial V}{\partial\delta}\right)_{\delta_0}\right)\dot{\delta}_0$$

and a corresponding equation for $\dot{\delta}_1$ give us $\dot{\delta}_0$ and $\dot{\delta}_1$ and so $\dot{\lambda}$ and $\dot{\epsilon}$.

The equation

$$t_0 = t - \int_0^u \frac{B\,du}{\sqrt{K(\delta)}}$$

where

$$\left\{ 2B \left(H - \frac{1}{2A} \dot{\Psi}^2 - V(\delta, t) \quad \sin^2 \delta - (\Phi - \Psi \cos \delta)^2 \right\} =$$

$$(\cos \delta_0 - \cos \delta)(\cos \delta - \cos \delta_1) \, K(\delta)$$

gives us

$$\dot{t}_0 = 1 - \frac{B}{\sqrt{K(\delta)}} \, \dot{u} + \int_0^u \frac{B}{2(K(\delta))^{3/2}} \frac{\delta}{\delta t} \left[K(\delta) \right] \, du$$

where

$$\frac{\delta}{\delta t}\left[K(\delta)\right] \qquad \frac{\delta}{\delta t}\left[K(\delta)\right] \quad \text{means} \quad \frac{\partial K}{\partial V} \frac{\partial V}{\partial t} + \frac{\partial K}{\partial \Psi} \dot{\Psi} + \frac{\partial K}{\partial \Phi} \dot{\Phi} + \frac{\partial K}{\partial \delta_0} \dot{\delta}_0 +$$

$$\frac{\partial K}{\partial \delta_1} \dot{\delta}_1 + \frac{\partial K}{\partial H} \dot{H}$$

evaluated after δ has been expressed in terms of δ_0, δ_1, and u.

Hence $\dot{t}_0 = \frac{B}{\sqrt{K(\delta)}} \left(\frac{\partial u}{\partial \delta} \frac{F_2}{mv} - \frac{\partial u}{\partial \delta_0} \dot{\delta}_0 - \frac{\partial u}{\partial \delta_1} \dot{\delta}_1 \right) +$

$$\int_0^u \frac{B}{2(K(\delta))^{3/2}} \frac{\delta}{\delta t} \left[K(\delta) \right] \, du$$

where although the separate terms of the first bracket become infinitely large as $\delta \rightarrow \delta_0$ or $\delta \rightarrow \delta_1$, their sum remains finite and small.

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In a similar way $\dot{\phi}_0$ and $\dot{\psi}_0$ can be found.

Thus the time rates of change of H, Φ, Ψ, t_0, ϕ_0, and ψ_0, are found as linear combinations of F_2, F_3, G_A, G_B, $G_3 + \dfrac{\partial V}{\partial \delta}$, and $\dfrac{\partial V}{\partial t}$, with coefficients that involve H, Φ, Ψ, t_0, ϕ_0, and ψ_0, and t. If these equations, and those giving $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$, $\dot{a}$, and $\dot{\theta}$, are averaged over a period of the solution of the reduced equations, we get the secular equations that give the first approximation to the changes of H, Φ, Ψ, t_0, ϕ_0, and ψ_0. If we have been able to choose V so that these changes are small over a period, this will be a useful method of solution. For a symmetrical projectile the right hand sides of the secular equations will not involve t_0 and ψ_0 and will involve ϕ_0 only in the terms involving gravitational forces.

3.5 Stability of Yawing Motion

The kind of stability we are concerned with is that δ_0 and δ_1 should, if small, remain small, if large, become small, regarding all the variables except ϕ, δ, Φ, and Δ, as practically constant. Taking $V(\delta)$ so that $\dfrac{\partial V}{\partial \delta} + G_3$ is small, we look first at reduced equations which give a conservative system with H and Φ constant.

In order that δ_0 and δ_1 can be small, it is necessary that for the value of Φ for which ε can be small, namely Ψ, the part of H depending on δ, which reduces to

$$\frac{\Psi^2}{2B} \; \frac{1 - \cos \delta}{1 + \cos \delta} + V(\delta)$$

should have a minimum at $\delta = 0$. Then δ_1 must initially be less than the first maximum of this same expression, if there is to be stability.

Next we must look at the secular equations for $\dot{H}$ and $\dot{\Phi}$. These, if the motion with small yaw is stable, give, for some region near $H = V(0)$, $\Phi = \Psi$, motion tending asymptotically to these values.

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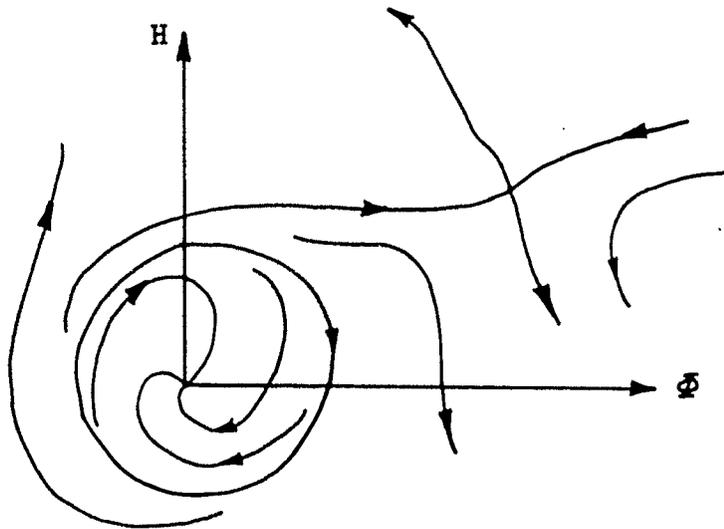
This region is the region of initial values for which the yawing motion is to be regarded as stable.

It should be remarked that this 'secular stability' does not exclude the possibility that in higher order approximation resonance may exist leading to unstable motions.

To determine the region of stability we solve the ordinary differential system giving the secular change in H and $\bar{\Phi}$,

$$\dot{H} = f(H, \bar{\Phi})$$

$$\dot{\bar{\Phi}} = g(H, \bar{\Phi})$$

starting in various directions from points near the point of equilibrium $H = V(o)$, $\bar{\Phi} = \bar{\Psi}$, and going backwards in time, and these solutions mark out the region.

Some information may be obtained by examining the equations

$$f(H, \bar{\Phi}) = o$$

$$g(H, \bar{\Phi}) = o$$

for other points of equilibrium, but there is no easier way of determining the existence of an asymptotic periodic solution that may separate the stable and unstable regions than by making a numerical integration. Such as integration for a single starting value may be sufficient.

Stable periodic solutions may also exist.

We have, in fact, to deal with the simplest typical problem of non-linear dynamics.

If we wish to take into account the effects of slow changes in the coefficients and in $\bar{\Psi}$, we should work rather in terms of λ and ϵ than in terms of H and $\bar{\Phi}$, since it is the size of λ and ϵ in which we are really interested.

Part IV. Detailed Approximate Theory

4.1 A Special Form of the Reduced Equations

Ω

We may choose such a form for $V(\delta)$ that $VK(\delta)$ shall be constant, Ω say.

Thus $V(\delta) = H - \frac{1}{2A}\Psi_o^2 + \frac{1}{2B}\Psi_o^2 - \frac{1}{2B}\Omega^2$

μ, ν $+ \dfrac{\left\{\Omega^2(\cos^2\lambda + \cos^2\epsilon) - \Phi_o^2 - \Psi_o^2\right\} + 2(\Phi_o\Psi_o - \Omega^2\cos\lambda\cos\epsilon)\cos\delta}{2B\sin^2\delta}$

If, then we take

$$V(\delta) = V_o - \frac{4\mu}{1 + \cos\delta} + \frac{4\nu}{1 - \cos\delta}$$

we have $\Omega^2(\cos\lambda - \cos\epsilon)^2 = (\Psi_o - \Phi_o)^2 + 16B\nu,$

$$\Omega^2(\cos\lambda + \cos\epsilon)^2 = (\Psi_o + \Phi_o)^2 - 16B\mu,$$

and $\dfrac{\Omega^2}{2B} = H - \frac{1}{2A}\Psi_o^2 + \frac{1}{2B}\Psi_o^2 - V_o$.

The solution of the reduced equations is now, for $\epsilon < \lambda$,

$\triangle = \Omega\sin\lambda\sin v$

$\Phi = \Phi_o$

$\Psi = \Psi_o$

$t_o = t - Bu/\Omega$

$\phi_o = \phi - \dfrac{\Phi_o\cos\lambda - \Psi_o\cos\epsilon}{(\cos^2\epsilon - \cos^2\lambda)}\dfrac{v}{\Omega} - \dfrac{(\Phi_o\cos\epsilon - \Psi_o\cos\lambda)}{(\cos^2\epsilon - \cos^2\lambda)}\dfrac{(\pi - w)}{\Omega}$

$\psi_o = \psi - \left(\dfrac{B}{A} - 1\right)\dfrac{\Psi_o}{\Omega}u - \left(\dfrac{\Psi_o\cos\lambda - \Phi_o\cos\epsilon}{\cos^2\epsilon - \cos^2\lambda}\right)\dfrac{v}{\Omega} -$

$\left(\dfrac{\Psi_o\cos\epsilon - \Phi_o\cos\lambda}{\cos^2\epsilon - \cos^2\lambda}\right)\dfrac{(\pi - w)}{\Omega}.$

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Notice that for $V(\delta)$ constant, this reduces to the usual Eulerian nutation of a symmetrical rotator, while by taking $\nu = 0$, we obtain motion corresponding to overturning moment

$$\frac{4\,\mu\,\sin\delta}{(1+\cos\delta)^2}\quad,$$

which will serve as a good reduced motion for investigating stability with large yaw.

.In general, we have a motion with mean yaw λ, amplitude of nutation ϵ and frequency of nutation $\Omega/2\pi B$.

The effects of replacing the overturning moment by $\dfrac{4\mu\,\sin\delta}{(1+\cos\delta)^2}$

will not show up in our approximate system of secular equations; they show up immediately in the equation for t_o which now takes the form

$$\dot{t}_o = \frac{B}{\Omega\,\sin\lambda\,\sin v}\left(\frac{F_2}{Mv} + \dot{\epsilon}\cos w + \dot{\lambda}\cos v\right)\quad,$$

containing, when $\dot{\epsilon}$ and $\dot{\lambda}$ are expressed in terms of the forces, the

term $\dfrac{B}{\Omega^2}$ $(\cot\epsilon\cos w + \cot\lambda\cos v)$ $\left(G_3 - \dfrac{4\mu\,\sin\delta}{(1+\cos\delta)^2}\right)$.

The leading term in $G_3 = M$ is $\rho\,v^2\,d^3\,\sin\delta\,K_M$, and we should take μ so that

$$(\cot\epsilon\cos w + \cot\lambda\cos v)\left(\rho\,v^2 d^3\,\sin\delta\,K_M - \frac{4\mu\,\sin\delta}{(1+\cos\delta)^2}\right)$$

vanishes when averaged over the reduced motion, after inserting the actual variation of K_M with δ.

If K_M does not depend on δ, this gives (see Appendix)

$$2\cos\epsilon\cos\lambda\,\rho\,v^2\,d^3\,K_M = \frac{4\mu}{\cos\epsilon + \cos\lambda}$$

In this case $\dfrac{\Psi^2}{2B}\dfrac{1-\cos\delta}{1+\cos\delta} + V(\delta)$ has no maximum, and the only

preliminary condition is $\Psi^2 > 4\,B\,\mu$. that is, that the stability factor for small yaw is greater than unity. If, however, K_M

increases with δ faster than $1/(1+\cos\delta)^2$, a finite stable range of δ may be determined.

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4.2 An Approximate Form for the Secular Equations

We shall systematically neglect terms of higher order than the first in Δ and Σ in the aerodynamic force and torque components, so that, for instance, we take

$$J = \rho \, v \, d^4 \, \sin \delta \, \frac{\Psi}{A} K_J - \rho \, d^4 \, \frac{\Delta}{B} \, \frac{\Psi}{A} \, K_{XT} + \rho \, d^4 \, v \frac{\Sigma}{B} K_H \, ,$$

with the coefficients taken to be functions of M and R, and even functions of $\frac{d}{v} \frac{\Psi}{A}$ and δ, but not to depend on Δ and Σ.

We shall also omit the gravitational terms.

Since in the secular equations terms linear in Δ will vanish on the average, we obtain

$$\dot{H} = \rho \, v \, d^4 \, \overline{\left(\frac{\Delta}{B}\right)^2} K_H + \rho \, v \, d^4 \, \overline{\frac{\Psi}{A} \frac{\Sigma}{B} \sin \delta} \, K_J + \rho \, v \, d^4 \, \overline{\left(\frac{\Sigma}{B}\right)^2} K_H$$

$$- \rho \, v \, d^4 \overline{\left(\frac{\Psi}{A}\right)^2} K_I - \rho \, v \, d^4 \, \overline{\frac{\Psi}{A} \frac{\Sigma}{B} \sin \delta} \, K_{I \Sigma} - \rho \, \frac{v^2 d^2}{Mv} \, \overline{\sin \delta \frac{\partial v}{\partial \delta}} \, K_L$$

$$- \frac{\rho \, d^4}{Mv} \, \overline{\frac{\Psi}{A} \frac{\Sigma}{B} \frac{\partial v}{\partial \delta}} K_{XF} + \overline{\frac{\partial v}{\partial t}}$$

$$\dot{\Phi} = - \rho \, v \, d^4 \, \frac{\Psi}{A} \overline{\cos \delta \, K_I} - \rho \, v \, d^4 \, \overline{\frac{\Sigma}{B} \cos \delta \sin \delta \, K_{I \Sigma}}$$

$$- \rho \, v \, d^4 \, \frac{\Psi}{A} \overline{\sin^2 \delta \, K_J} - \rho \, v \, d^4 \, \overline{\frac{\Sigma}{B} \sin \delta \, K_H} + \frac{\rho \, v^2 d^2}{Mv} \overline{\left(\Psi - \Phi \cos \delta\right) K_L} +$$

$$\frac{\rho \, d^4}{Mv} \, \frac{\Psi}{A} \overline{\frac{\left(\Psi - \Phi \cos \delta\right)}{\sin \delta} \frac{\Sigma}{B} K_{XF}} + \frac{\rho \, d^4}{Mv} \, \frac{\Psi}{A} \overline{\frac{\Delta^2}{B} K_{XF}}$$

and

$$\dot{\Psi} = - \rho \, v \, d^4 \, \frac{\Psi}{A} \overline{K_I} - \rho \, v \, d^4 \, \overline{\frac{\Sigma}{B} \sin \delta \, K_{I \Sigma}} \, .$$

These averages should be taken over the reduced motion after inserting the actual variations of the coefficients with δ. This may perhaps be done by expressing them approximately as polynomials in δ.

We will now have $\dot{H}$, $\dot{\Phi}$, and $\dot{\Psi}$, given in terms of Φ, Ψ, λ, ϵ, and Ω, as well as μ, and we can use the equations

$$\cos \lambda - \cos \epsilon = (\Phi - \Psi) / \Omega$$

$$\cos \lambda + \cos \epsilon = \sqrt{(\Psi + \Phi)^2 - 16 \, B \, \mu} / \Omega$$

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where $\Omega = \sqrt{2 B (H - \frac{1}{2A} \Psi^2 + \frac{1}{2B} \Psi^2 - V_0)}$

to obtain $\frac{d}{dt} \cos \lambda$ and $\frac{d}{dt} \cos \epsilon$ in terms of λ, ϵ, Ψ, and μ. The stability problem then involves the corresponding problem in non-linear dynamics, regarding Ψ and μ as slowly changing.

If the aerodynamic coefficients do not depend on δ we obtain (see Appendix)

$$\dot{H} =$$

$$\rho v d^4 \frac{\Omega^2}{B^2} \frac{\sin^2 \lambda}{} \frac{1 - \cos \epsilon}{\sin^2 \lambda} K_H + \rho v d^4 \frac{\Psi}{A} \left(\frac{\Psi}{B} \cos\epsilon \cos \lambda - \frac{\Phi}{B} \right) K_J$$

$$+ \rho v d^4 \left\{ \frac{\Psi^2}{B^2} \left(\frac{\cos \epsilon}{\cos^2 \epsilon - \cos^2 \lambda} - 1 \right) - \frac{2 \Psi\Phi}{B^2} \frac{\cos \lambda}{\cos^2 \epsilon - \cos^2 \lambda} + \frac{\Phi^2}{B^2} \frac{\cos \epsilon}{\cos^2 \epsilon - \cos^2 \lambda} \right\} K_H$$

$$- \rho v d^4 \left\{ \frac{\Psi^2}{A^2} K_I - \rho v d^4 \frac{\Psi}{A} \left(\frac{\Psi}{B} \cos\epsilon \cos \lambda - \frac{\Phi}{B} \right) K_{I\Sigma} + \frac{\rho v^2 d^2}{Mv} 4\mu \left(\frac{2}{\cos\epsilon + \cos \lambda} - 1 \right) K_L \right.$$

$$+ \rho \frac{d^4}{Mv} \frac{\Psi}{A} 4\mu \left(\frac{\Psi}{B} \frac{\cos^2 \epsilon + \cos^2 \lambda + \cos\epsilon \cos \lambda - 1}{(\cos\epsilon + \cos \lambda)^3} - \frac{\Phi}{B} \frac{1 + \cos \epsilon \cos \lambda}{(\cos \epsilon + \cos \lambda)^3} \right) K_{XF} - \frac{4\dot{\mu}}{\cos\epsilon + \cos\lambda}$$

$$\dot{\Phi} =$$

$$- \rho v d^4 \frac{\Psi}{A} \cos\epsilon \cos \lambda K_I - \rho v d^4 \left(\frac{\Psi}{A} \left(\cos^2 \epsilon \cos^2 \lambda + \frac{1}{2} \sin^2 \epsilon \sin^2 \lambda \right) - \frac{\Phi}{B} \cos\epsilon \cos\lambda \right) K_{I\Sigma}$$

$$- \rho v d^4 \frac{\Psi}{A} \left(1 - \cos^2 \epsilon \cos^2 \lambda - \frac{1}{2} \sin^2 \epsilon \sin^2 \lambda \right) K_J - \rho v d^4 \left(\frac{\Psi}{B} \cos\epsilon \cos \lambda - \frac{\Phi}{B} \right) K_H$$

$$+ \frac{\rho v^2 d^2}{Mv} \left(\Psi - \Phi \cos\epsilon \cos \lambda \right) K_L + \frac{\rho d^4}{Mv} \frac{\Psi}{A} \left[\frac{\Psi^2 + \Phi^2}{B} \frac{\cos \lambda}{\cos^2 \epsilon - \cos^2 \lambda} \right.$$

$$- \frac{\Phi \Psi}{B} \left(\frac{2 \cos \epsilon}{\cos^2 \epsilon - \cos^2 \lambda} - 1 \right)$$

$$\left. + \frac{\rho d^4}{Mv} \frac{\Psi}{A} \Omega^2 \frac{\sin^2 \lambda}{B} \frac{1 - \cos \epsilon}{\sin^2 \lambda} \right] K_{XF}$$

$$\dot{\Psi} = - \rho v d^4 \frac{\Psi}{A} K_I - \rho v d^4 \left(\frac{\Psi}{B} \cos\epsilon \cos \lambda - \frac{\Phi}{B} \right) K_{I\Sigma}$$

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4.3 The Special Case of Small Yaw

The equation $\Omega^2 (\cos\lambda - \cos\epsilon)^2 = (\dot\Psi - \dot\Phi)^2$ ensures that $\dot H$, $\dot\Phi$, and $\dot\Psi$, are not large, on account of the denominator $\cos\lambda - \cos\epsilon$, when the yaw is small. Using this equation and keeping only terms of order $\dot\Psi - \dot\Phi$ and ϵ^2 and λ^2 we obtain

$$\dot H = \rho\, v\, d^4 \left\{ \frac{\Omega^2}{B^2}\; \frac{\lambda^2+\epsilon^2}{4} + \frac{\dot\Psi^2}{B^2}\; \frac{\epsilon^2+\lambda^2}{4} - \frac{\dot\Psi(\dot\Psi-\dot\Phi)}{B^2}\right\} K_H$$

$$+ \rho\, v\, d^4\, \frac{\dot\Psi}{A}\left(\frac{\dot\Psi-\dot\Phi}{B} - \frac{\dot\Psi}{B}\;\frac{(\epsilon^2+\lambda^2)}{2}\right) K_J - \rho\, v\, d^4\, \frac{\dot\Psi^2}{A^2} K_I$$

$$- \rho\, v\, d^4\, \frac{\dot\Psi}{A}\left(\frac{\dot\Psi-\dot\Phi}{B} - \frac{\dot\Psi}{B}\;\frac{(\epsilon^2+\lambda^2)}{2}\right) K_{I\Sigma} + \frac{\rho\, v^2 d^2}{Mv}\, \mu\,(\epsilon^2+\lambda^2)\, K_L$$

$$+ \rho\, \frac{d^4}{Mv}\, \frac{\dot\Psi}{A}\, \mu\left(\frac{\dot\Psi-\dot\Phi}{B} - \frac{\dot\Psi}{B}\;\frac{(\epsilon^2+\lambda^2)}{2}\right) K_{XF} - 2\dot\mu\left(1 + \frac{\epsilon^2+\lambda^2}{4}\right)$$

$$\dot\Phi = -\rho\, v\, d^4\, \frac{\dot\Psi}{A}\left[1 - \frac{(\epsilon^2+\lambda^2)}{2}\right] K_I - \rho\, v\, d^4\left[\frac{(\dot\Psi-\dot\Phi)}{B} - \frac{\dot\Psi}{B}\left(\frac{\epsilon^2+\lambda^2}{2}\right)\right] K_{I\Sigma}$$

$$- \rho\, v\, d^4\, \frac{\dot\Psi}{A}\,(\epsilon^2+\lambda^2)\, K_J - \rho\, v\, d^4\left[\left(\frac{\dot\Psi-\dot\Phi}{B} - \frac{\dot\Psi}{B}\left(\frac{\epsilon^2+\lambda^2}{2}\right)\right)\right] K_H$$

$$+ \frac{\rho\, v^2\, d^2}{Mv}\left[\dot\Psi-\dot\Phi + \dot\Psi\left(\frac{\epsilon^2+\lambda^2}{2}\right)\right] K_L + \frac{\rho\, d^4}{Mv}\, \frac{\dot\Psi}{A}\left[\frac{\Omega^2}{4B}(\epsilon^2+\lambda^2) - \frac{\dot\Psi^2}{B}\left(\frac{\epsilon^2+\lambda^2}{4}\right)\right] K_{XF}$$

$$\dot\Psi = -\rho\, v\, d^4\, \frac{\dot\Psi}{A}\, K_I - \rho\, v\, d^4\left[\frac{\dot\Psi-\dot\Phi}{B} - \frac{\dot\Psi}{B}\,\frac{1}{2}\left(\epsilon^2+\lambda^2\right)\right] K_{I\Sigma}$$

For small yaw, λ and ϵ are small, and

$$\Omega^2 \approx \dot\Psi^2 - 4B\mu$$

$$\mu \approx \rho\, v^2\, d^3\, K_M$$

while $\dot\Psi - \dot\Phi \approx \Omega\left(\frac{\lambda^2 - \epsilon^2}{2}\right)$

and $H - \frac{\dot\Psi^2}{2B} + 2\mu - V_0 \approx \frac{\Omega^2}{4B}(\lambda^2+\epsilon^2) - \frac{\dot\Psi}{2B}\left(\dot\Psi-\dot\Phi\right)$

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These are of the first order in λ^2 and. ϵ^2, and so are their time rates of change.

In fact, if we write for brevity,

$$h = \frac{-\rho\, v\, d^4}{B}\, K_H$$

$$k = \frac{\rho\, v\, d^2}{Mv}\, K_L$$

$$\ell = \frac{\rho\, v\, d^4}{A}\, K_I$$

$$\gamma = \frac{\rho\, v\, d^4}{A}\, K_J$$

$$\chi = \frac{\rho\, d^4}{Mv\, A}\, K_{XF}$$

we obtain

$$\dot{\underline{\Psi}} - \dot{\phi} = \Psi\left\{\gamma + \tfrac{1}{2} h - \tfrac{1}{2} k - \tfrac{1}{2}\ell + \mu\chi\right\}\left(\lambda^2 + \epsilon^2\right) - \Omega\left(\tfrac{1}{2} h + \tfrac{1}{2} k\right)\left(\lambda^2 - \epsilon^2\right)$$

$$\dot{H} - \frac{\Psi\,\dot{\Psi}}{B} + 2\dot{\mu} = \frac{1}{2B}\left\{\tfrac{1}{2} k\left(\Psi^2 - \Omega^2\right) - \tfrac{1}{2} h\left(\Psi^2 + \Omega^2\right) - \Psi^2\left(\gamma + \mu\chi\right)\right.$$

$$\left. - B\,\dot{\mu}\right\}\left(\lambda^2 + \epsilon^2\right) + \frac{\Omega}{2B}\,\Psi\left(h + \gamma + \chi\mu\right)\left(\lambda^2 - \epsilon^2\right) ,$$

while, keeping only the leading term,

$$\dot{\Psi} = -\ell\,\Psi .$$

From these equations we find, writing $\sigma = \dfrac{\Omega}{\Psi}$, $\dfrac{\dot{\sigma}}{\sigma} = \dfrac{\dot{\Omega}}{\Omega} - \dfrac{\dot{\Psi}}{\Psi}$,

so that $\dfrac{\dot{\Omega}}{\Omega} = \dfrac{\dot{\sigma}}{\sigma} - \ell$, while $B\,\dot{\mu} = -\tfrac{1}{2}\Psi^2\ell + \tfrac{1}{2}\Omega^2\left(\ell - \dfrac{\dot{\sigma}}{\sigma}\right) ,$

$$\frac{d}{dt}\left(\lambda^2 - \epsilon^2\right) = \frac{1}{\sigma}\left(2\gamma + h - k - \ell + 2\chi\mu\right)\left(\lambda^2 + \epsilon^2\right) - \left(h + k - \ell + \frac{\dot{\sigma}}{\sigma}\right)\left(\lambda^2 - \epsilon^2\right)$$

$$\frac{d}{dt}\left(\lambda^2 + \epsilon^2\right) = -\left(h + k - \ell + \frac{\dot{\sigma}}{\sigma}\right)\left(\lambda^2 + \epsilon^2\right) + \frac{1}{\sigma}\left(2\gamma + h - k - \ell + 2\chi\mu\right)\left(\lambda^2 - \epsilon^2\right)$$

so that

$$\frac{d}{dt}\lambda^2 = -\left\{\left(h + k - \ell + \frac{\dot{\sigma}}{\sigma}\right) - \frac{1}{\sigma}\left(2\gamma + h - k - \ell + 2\chi\mu\right)\right\}\lambda^2$$

$$\frac{d}{dt} \epsilon^2 = -\left\{ \left(h + k - \ell + \frac{\dot{\sigma}}{\sigma} \right) + \frac{1}{\sigma} \left(2\gamma + h - k - \ell + 2\chi\mu \right) \right\} \epsilon^2$$

and the condition of stability, for real σ, is that both the coefficients on the right hand side should be positive.
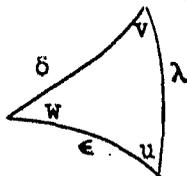
If u, v, and w, are the angles of a spherical triangle with sides δ, ϵ, and λ, we have the following formulas.

$$\frac{\sin u}{\sin \delta} = \frac{\sin v}{\sin \epsilon} = \frac{\sin w}{\sin \lambda}$$

$$\tan \frac{u}{2} = \sqrt{\frac{\cos(\lambda - \epsilon) - \cos \delta}{\cos \delta - \cos(\lambda + \epsilon)}}$$

$$\tan \frac{v}{2} = \sqrt{\frac{\cos(\delta - \lambda) - \cos \epsilon}{\cos \epsilon - \cos(\delta + \lambda)}}$$

$$\tan \frac{w}{2} = \sqrt{\frac{\cos(\epsilon - \delta) - \cos \lambda}{\cos \lambda - \cos(\epsilon + \delta)}}$$

$$\cos \delta = \cos \epsilon \; \cos \lambda + \sin \epsilon \; \sin \lambda \; \cos u$$

$$\cos \epsilon = \cos \lambda \; \cos \delta + \sin \lambda \; \sin \delta \; \cos v$$

$$\cos \lambda = \cos \delta \; \cos \epsilon + \sin \delta \; \sin \epsilon \; \cos w$$

$$\cos u = - \cos v \cos w + \sin v \sin w \; \cos \delta$$

$$\cos v = - \cos w \cos u + \sin w \sin u \; \cos \epsilon$$

$$\cos w = - \cos u \cos v + \sin u \sin v \; \cos \lambda$$

$$\cos v \; \sin \delta = \sin \lambda \; \cos \epsilon - \cos \lambda \sin \epsilon \cos u$$

$$\cos v \; \sin \lambda = \sin \delta \; \cos \epsilon - \cos \delta \sin \epsilon \cos w$$

$$\cos w \; \sin \delta = \sin \epsilon \; \cos \lambda - \cos \epsilon \sin \lambda \cos u$$

$$\cos w \; \sin \epsilon = \sin \delta \; \cos \lambda - \cos \delta \; \sin \lambda \cos v$$

$$\cos u \; \sin \lambda = \sin \epsilon \; \cos \delta - \cos \epsilon \sin \delta \; \cos w$$

$$\cos u \; \sin \epsilon = \sin \lambda \; \cos \delta - \cos \lambda \sin \delta \; \cos v$$

$$\cos \epsilon \; \sin w = \sin u \; \cos v + \cos u \sin v \; \cos \lambda$$

$$\cos \epsilon \; \sin u = \sin w \; \cos v + \cos w \sin v \; \cos \delta$$

$$\cos \lambda \ \sin u = \sin v \ \cos w + \cos v \ \sin w \cos \delta$$

$$\cos \lambda \ \sin v = \sin u \ \cos w + \cos u \ \sin w \cos \epsilon$$

$$\cos \delta \ \sin v = \sin w \ \cos u + \cos w \ \sin u \cos \epsilon$$

$$\cos \delta \ \sin w = \sin v \ \cos u + \cos v \ \sin u \cos \lambda$$

$$\dot{u} \sin v \sin \lambda = \dot{\delta} - \dot{\epsilon} \cos w - \dot{\lambda} \cos v$$

$$\dot{v} \sin w \sin \delta = \dot{\epsilon} - \dot{\lambda} \cos u - \dot{\delta} \cos w$$

$$\dot{w} \sin u \sin \epsilon = \dot{\lambda} - \dot{\delta} \cos v - \dot{\epsilon} \cos u$$

The above formulas may be used to average expressions with respect to u. In particular, since

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{a + b \cos u} = \frac{1}{\sqrt{a^2 - b^2}}$$

and
$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{(a + b \cos u)^2} = \frac{a}{(a^2 - b^2)^{3/2}}$$

we find, for $\epsilon < \lambda$, the following averages:

$$\overline{\cos \delta} = \cos \epsilon \ \cos \lambda$$

$$\overline{\cos^2 \delta} = \cos^2 \epsilon \ \cos^2 \lambda + \frac{1}{2} \sin^2 \epsilon \ \sin^2 \lambda$$

$$\overline{\frac{1}{1 + \cos \delta}} = \frac{1}{\cos \epsilon + \cos \lambda}$$

$$\overline{\frac{1}{1 - \cos \delta}} = \frac{1}{\cos \epsilon - \cos \lambda}$$

$$\overline{\frac{1}{\sin^2 \delta}} = \frac{\cos \epsilon}{\cos^2 \epsilon - \cos^2 \lambda}$$

$$\overline{\frac{\cos \delta}{\sin^2 \delta}} = \frac{\cos \lambda}{\cos^2 \epsilon - \cos^2 \lambda}$$

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$$\overline{\sin^2 v} = \frac{1 - \cos \epsilon}{\sin^2 \lambda}$$

$$\overline{\frac{\sin^2 \delta}{(1 + \cos \delta)^2}} = \frac{2}{\cos \epsilon + \cos \lambda} - 1$$

$$\overline{\frac{1}{(1 + \cos \delta)^2}} = \frac{1 + \cos \epsilon \; \cos \lambda}{(\cos \epsilon + \cos \lambda)^3}$$

$$\overline{\frac{\cos \delta}{(1 + \cos \delta)^2}} = \frac{\cos^2 \epsilon + \cos^2 \lambda + \cos \epsilon \; \cos \lambda - 1}{(\cos \epsilon + \cos \lambda)^3}$$

$$\overline{\cos w \sin \delta} = \sin \epsilon \; \cos \lambda$$

$$\overline{\cos v \sin \delta} = \cos \epsilon \; \sin \lambda$$

$$\overline{\frac{\cos w \sin \delta}{(1 + \cos \delta)^2}} = \frac{\sin \epsilon}{(\cos \epsilon + \cos \lambda)^2}$$

$$\overline{\frac{\cos v \sin \delta}{(1 + \cos \delta)^2}} = \frac{\sin \lambda}{(\cos \epsilon + \cos \lambda)^2}$$

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